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INTERACTION OF CRACKS WITH RIGID INCLUSIONS IN LONGITUDINAL SHEAR DEFORMATION

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ABSTRACT

The Interaction of a crack with rigid circular cylindrical inclusions is considered for the case of longitudinal shear deformation. General representations of the solutions for a radial crack near a single and midway between two inclusions are given. The particular case of uniform shearing stress applied at infinity is discussed in detail.

Expressions for the crack tip stress intensity factor K_3 are derived and it is shown that $K_3 = 0$ for a crack tip at the inclusion boundary, provided that the crack is radial. Generalization of the results to any number of inclusions with common line of centers is indicated.

1. INTRODUCTION

The theoretical literature dealing with fracture of composite materials is rather limited with rigorous analytical investigations being restricted to extremely idealized models. The most common idealization consists of replacing the heterogeneous composite by an "equivalent" homogeneous anisotropic medium. By performing analytical fracture studies on the equivalent homogeneous solid and comparing the theoretical results with experimental data, values of K_c (crack tip stress intensity factor) and G_c (energy release rate) can be obtained, which characterize the macroscopic fracture behavior of the composite. This approach has met with considerable success.

Considering that the composite is actually a heterogeneous solid consisting of homogeneous phases possessing individual values of the fracture parameters, it is natural to inquire whether the K_c value of the composite can be predicted from those of the constituents. To provide an answer to this question, one has to perform a rigorous analysis that takes the heterogeneity of the composite into account.

In general, the mathematical difficulties that are encountered are sufficiently great to force one to make severe approximations. A number of analyses, based on approximate models, are available [1-6]. The most common approximation has been to study the interaction of a crack with a single circular inclusion. This problem has been solved exactly by Tamate [3] and Plato [4] and approximately by Sih et al. [1], Smith [5] and Atkinson [6]. When the transition is made to models resembling the microstructure of the composite more closely, only approximate analyses have been attempted [2].

In the present work, some exact analyses are presented for a limiting case of a fiber reinforced composite. The particular limit that is considered is the case in which the fiber shear modulus is much greater than that of the matrix. In this case it is reasonable to assume the fibers to be rigid. Furthermore, the composite is assumed to consist of aligned circular cylindrical fibers that are perfectly bonded to the matrix. The crack is assumed to lie in a plane parallel to the fibers and to extend to infinity in the fiber direction. The analysis for the general case is synthesized from a number of elementary cases of ever increasing complexity.

The structure of the solution for a crack in a homogeneous solid under longitudinal shear loading is examined in Section 2 and a general representation satisfying the boundary conditions at the crack is constructed. This general representation is used in Section 3 to construct a general representation of the solution for the problem of a (radial) crack near a rigid circular inclusion. The general representation of the solution for a crack lying on the line of centers between two rigid inclusions is found in Section 4. In Section 5, the results are generalized to a crack interacting with any number of inclusions, provided that the crack and inclusion centers lie on the same line. Since the general representation is also indicated in this case, it can be argued that the proper choice of the arbitrary function in the representation will permit one to solve the problem of a crack interacting with an arbitrary array of inclusions.

Furthermore, upon calculating the value of the crack tip stress intensity factor for each of the examples considered, it is observed that the rigid inclusions tend to decrease its magnitude. This trend has been

observed in references [1-6]. It is further observed that the crack tip stress intensity factor is identically zero when the crack tip reaches the rigid inclusion. Since this is observed for all the cases examined, it can be concluded that the macroscopic fracture toughness of the composite may not be related to the fracture toughness values of the constituents.

2. CRACK IN HOMOGENEOUS SOLID

Consider an unbounded solid containing a crack which occupies the region $|x| < c$, $-\infty < z < \infty$, $y = 0$. If the solid is loaded by shearing forces, which are functions of x and y only, parallel to the z-axis, the resulting deformation will be of longitudinal shear (or anti-plane) type. In this case, one has

$$u = v = 0 \ , \quad w = w(x,y) \ , \tag{1}$$

where u, v, w are displacements along the coordinate axes. The nonzero displacement and stress composnents are given, in terms of an analytic function $F(z)$ of the complex variable $z = x + iy$,* by

$$\mu \, w(x,y) = \mathrm{Re}\,\{F(z)\} \ , \tag{2}$$

$$\sigma_{xz} - i\,\sigma_{yz} = \frac{dF}{dz} \ , \tag{3}$$

where Re {.} is used to denote the real part of the function in the brackets and μ is the shear modulus. In terms of polar coordinates,

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* Henceforth, z will be used to denote the complex variable $x = iy$.

(3) becomes

$$\sigma_{rz} - i\,\sigma_{\theta z} = e^{i\theta}\,\frac{dF}{dz} \quad . \tag{4}$$

The boundary condition, to be satisfied on the crack surface, is

$$\sigma_{yz} = 0 \quad \text{on} \quad y = 0 \;, \quad |x| \leq c \quad . \tag{5}$$

Given an arbitrary imposed stress field, defined by $F_\infty(z)$ in the homogeneous solid, it is required to determine the solution when the crack is present. In approaching the problem, it is convenient to decompose $F_\infty(z)$ into two parts, namely,

$$F_\infty(z) = g_1(z) + g_2(z) \quad , \tag{6}$$

where

$$2g_1(z) = F_\infty(z) - \bar{F}_\infty(z) \quad , \tag{7a}$$

$$2g_2(z) = F_\infty(z) + \bar{F}_\infty(z) \quad . \tag{7b}$$

Herein, the bar over only the function symbol denotes the complex conjugate of $F_\infty(z)$ with z being treated as though it were a real variable. For example, $\bar{F}_\infty(z) = -i + z$ when $F_\infty(z) = i + z$. It is readily verified that

$$\left.\begin{aligned} \frac{dg_1}{dz} &= \text{imaginary} \;, \\ \frac{dg_2}{dz} &= \text{real} \end{aligned}\right\} \quad \text{on } y = 0 \;. \tag{8}$$

Thus, $g_2(z)$ gives $\sigma_{yz} = 0$ on $y = 0$ and, hence, the boundary condition at the crack is automatically satisfied. This implies that if $F_\infty(z) = \bar{F}_\infty(z)$, the crack does not disturb the imposed stress field.

Let us now seek a solution of the problem in the case $F_\infty(z) = -\bar{F}_\infty(z)$. For convenience such an imposed str-ss field will be referred to as being of type I. For an imposed field of type I, the boundary conditions at the crack will be satisfied if a function $F(z)$ is found such that

$$\frac{dF}{dz} = \begin{cases} \text{real for } |x| \le c \\ \text{imaginary for } |x| \ge c \end{cases} \quad \text{on } y = 0 . \tag{9}$$

Such a function is given by

$$F(z) = f(z)\sqrt{z^2 - c^2} , \tag{10}$$

where $f(z)$ is an arbitrary function of type I. That (10) actually satisfies the boundary conditions at the crack is readily verified by substituting the shearing stresses corresponding to (10) into boundary condition (5). Since $f(z)$ is arbitrary, (10) can be considered to be a general representation of the solution of the crack problem for $F_\infty(z)$ of type I. It only remains to relate $f(z)$ in (10) to the imposed stress field. This can be readily done by insisting that solution (10) have the same behavior as $F_\infty(z)$ at infinity and at any other singularities of $F_\infty(z)$. In particular, if $F_\infty(z) = i\,P_n\,z^n$, the solution is given by

$$F(z) = i\,P_n\,z^{n-1}\sqrt{z^2 - c^2}\left\{1 + \sum_{m=1}^{M} \frac{(2m-1)!}{(m-1)!m!2^{2m-1}}\left(\frac{c}{z}\right)^{2m}\right\} \tag{11}$$

where

$$M = \begin{cases} n/2 & \text{for } n \text{ even} , \\ (n-1)/2 & \text{for } n \text{ odd} . \end{cases} \tag{12}$$

The structure of solution (11) is extremely simple. The finite series

in the brackets consists of the leading terms of the expansion of $[1 - (c/z)^2]^{-1/2}$ about the point at infinity, the series being terminated so that $F(z)$ does not become singular at the origin.

Finally, let us consider a special case of (11) which will prove useful in subsequent discussions. Letting $F_\infty(z) = - i P z$ (which corresponds to $\sigma_{xz}^\infty = 0$, $\sigma_{yz}^\infty = P$), we get

$$F(z) = - i P \sqrt{z^2 - c^2} \quad . \tag{13}$$

The crack tip stress intensity factor K_3^* corresponding to (13) is

$$K_3^* = P \sqrt{c} \quad . \tag{14}$$

This quantity will be needed in subsequent discussions. For a definition of the crack tip stress intensity factor, the reader is referred to [7].

3. RADIAL CRACK NEAR A RIGID INCLUSION

Let us now consider the interaction of a crack with a rigid inclusion of unit radius. Let the crack and inclusion be placed as shown in Figure 1. Referring to this figure, the boundary conditions to be satisfied at the crack and inclusion are

$$\sigma_{yz} = 0 \quad \text{on} \quad y = 0 \ , \ |x - a| \leq c \ , \tag{15}$$

$$w = 0 \quad \text{on} \quad |z| = 1 \ . \tag{16}$$

As in Section 2, given an arbitrary imposed stress field, defined by $F_\infty(z)$ in a homogeneous solid, it is required to determine the solution when the rigid inclusion and crack are present.

Before attacking this problem, let us recall a result given in [8,9] which can be stated as follows: given a stress field, defined by $f(z)$ which has no singularities inside the circle $|z| = 1$, in a homogeneous solid, the solution when a rigid inclusion of unit radius is present at the origin is given by

$$F(z) = f(z) - \bar{f}(z) \quad . \tag{17}$$

The information contained in (10) and (17) can be used to generate the solution of the problem formulated above. First of all, if $F_\infty(z) = \bar{F}_\infty(z)$, direct use of (17) gives

$$F(z) = F_\infty(z) - \bar{F}_\infty(z) \quad , \tag{18}$$

which satisfies the condition $F(z) = \bar{F}(z)$. Hence $F(z)$, defined by (18), is the desired solution.

If $F_\infty(z)$ is of type I, the situation is somewhat more complicated. The solution must simultaneously have the features of (10) and (17). Upon noting that the right hand side of (17) can be multiplied by a function that is real on $|z| = 1$ without violating (16) and that the product $\sqrt{(z - a)^2 - c^2}\sqrt{(1/z - a)^2 - c^2}$ is real on $|z| = 1$, it follows that the desired solution can be written in the form

$$F(z) = [h(z) - \bar{h}(1/z)]\sqrt{(z - a)^2 - c^2}\sqrt{(1/z - a)^2 - c^2}\,(a^2 - c^2)^{-1/2} \quad , \tag{19}$$

where $h(z)$ is a function of type I. That (19) is actually the desired representation can be verified by substituting (19) into the boundary conditions. Application of the boundary conditions at infinity (and at singularities of the applied stress field) will specify $h(z)$. In particular, taking $h(z) = - i\ P/2$ yields

$$F(z) = -i P \sqrt{(z-a)^2 - c^2} \sqrt{(1/z - a)^2 - c^2} \, (a^2 - c^2)^{-1/2} \quad , \qquad (20)$$

which is the solution for $\sigma^{\infty}_{xz} = 0$, $\sigma^{\infty}_{yz} = P$. In this case, the stress intensuty factor for the crack tip nearest to the rigid inclusion is

$$K_3 = P\sqrt{c} \sqrt{[1/(a-c) - a]^2 - c^2} \, (a^2 - c^2)^{-1/2} \qquad (21)$$

The ratio K_3/K_3^* is convenient for assessing the effect of the inclusion on the fracture behavior of the matrix. Dividing (21) by (14) gives

$$K_3/K_3^* = \sqrt{[1/(a-c) - a]^2 - c^2} \, (a^2 - c^2)^{-1/2} \quad . \qquad (22)$$

If the crack tip is at the inclusion interface, $c = a - 1$ and

$$K_3/K_3^* = 0 \quad ; \qquad (23)$$

that is, the stress intensity factor is zero for the crack tip at the rigid inclusion. Furthermore, $K_3 \leq K_3^*$ for all other admissible combinations of a and c.

It is readily verified that the stress intensity factor for the crack tip nearest the inclusion is given by

$$K_3/K_3^* = [h(a-c) - \bar{h}(1/(a-c))] \sqrt{[1/(a-c) - a]^2 - c^2} \, (a^2-c^2)^{-1/2} \qquad (24)$$

in the general case. Since the terms in the brackets are finite unless $h(z)$ is singular at $z = a - c$, we have the result that the stress intensity factor is zero for a crack terminating at a rigid inclusion no matter what the loading is.

4. CRACK BETWEEN TWO RIGID INCLUSIONS

Next, let us consider the interaction of a crack with two rigid inclusions of unit radius. Placing the crack and inclusions as shown in Figure 2, the boundary conditions to be satisfied at the crack and inclusions are

$$\sigma_{yz} = 0 \quad \text{on} \quad y = 0 \, , \, |x| \leq c \quad , \tag{25}$$

$$w = \text{rigid body displacement on } |z \pm a| = 1 \quad . \tag{26}$$

As in the previous sections, given an arbitrary imposed stress field, defined by $F_\infty(z)$ in a homogeneous solid, it is required to determine the solution when the two rigid inclusions and crack are present.

In solving this problem, the structure of the solution, presented in Section 3, will be used as a guide. Depending on the nature of $F_\infty(z)$, there are two cases to be considered. If $F_\infty(z) = \overline{F}_\infty(z)$, the crack does not influence the stress field and the problem reduces to the case of two rigid inclusions disturbing an applied stress field. This solution is readily found by the Schwarz alternating method [10,11]. It is given by

$$\begin{aligned} F(z) = F_\infty(z) - \sum_{n=0}^{\infty} \Big\{ & \overline{F}_\infty[a - \zeta_{2n}(z)] - \overline{F}_\infty[a - \zeta_{2n}(0)] \\ & + \overline{F}_\infty[\eta_{2n}(z) - a] - \overline{F}_\infty[\eta_{2n}(0) - a] \Big\} + \\ + \sum_{n=0}^{\infty} \Big\{ & F_\infty[a - \eta_{2n+1}(z)] - F_\infty[a - \eta_{2n+1}(0)] \\ & + F_\infty[\zeta_{2n+1}(z) - a] - F_\infty[\zeta_{2n+1}(0) - a] \Big\} \; , \end{aligned} \tag{27}$$

where

$$\zeta_{n+1}(z) = [2a - \zeta_n(z)]^{-1} \quad , \quad \eta_{n+1}(z) = [2a - \eta_n(z)]^{-1}$$
$$\zeta_0(z) = (a - z)^{-1} \quad , \quad \eta_0(z) = (a + z)^{-1} \quad ., \tag{28}$$

In the derivation of solution (27), the displacement of the point at the origin was specified to be the same as in the homogeneous solid. That solution (27) converges follows directly from the convergence proof for the Schwarz alternating method.

If $F_\infty(z)$ is of type I (i.e., $F_\infty(z) = - \overline{F}_\infty(z)$), the crack influences the stress field. As was done in Section 3, a solution will be sought in the form

$$F(z) = \sqrt{z^2 - c^2} \quad H(z)\, G(z) \quad , \tag{29}$$

where $H(z)$ is a type I solution in the absence of the crack and $G(z)$ is a function such that $G(z)\,(z^2 - c^2)^{1/2}$ is real on $|z \pm a| = 1$. It is readily verified that any solution of the form (27) can be used as $H(z)$ and that

$$G(z) = \prod_{n=0}^{\infty} \frac{\sqrt{[\zeta_n(z) - a]^2 - c^2}\,\sqrt{[\eta_n(z) - a]^2 - c^2}}{\sqrt{[\zeta_n(\infty) - a]^2 - c^2}\,\sqrt{[\eta_n(\infty) - a]^2 - c^2}} \tag{30}$$

Convergence of the infinite product can be established rather easily and, hence, it is not given here.

Even though solution (29) looks formidable, its structure is extremely simple. The functions $\zeta_n(z)$ and $\eta_n(z)$ are terminating continued fractions, resulting from successive inversions of z with respect to the rigid inclusions. The infinite product $G(z)$ consists

of crack type singularities, corresponding to cracks of diminishing length generated by inversion with respect to the inclusion boundaries. $H(z)$ is a general representation of the solution for the case of two rigid inclusions disturbing an applied stress field.

That (27) satisfies the boundary conditions can be easily verified. Upon noting that $H(z) = -\bar{H}(z)$ and $G(z) = \bar{G}(z)$, we have that $H(z)G(z)$ is a function of type I. Hence, upon using the results of Section 2, it follows that (29) satisfies boundary condition (25). To show that (29) satisfies (26), we note that $H(z)$ is imaginary on $|z \pm a| = 1$, while $G(z)\sqrt{z^2 - c^2}$ is real there. Thus, $F(z)$ given by (29) is imaginary on $|z \pm a| = 1$ and, hence, boundary conditions (26) are satisfied.

Since $H(z)$ in (29) is an arbitrary solution of the problem when the crack is absent, solution (29) can be considered to be a general representation of the solution for the problem being considered in this Section. Unfortunately, representation (29) is sufficiently complicated that only the simplest examples can be worked out in detail without overly tedious computations. For example, taking $H(z) = -iP$, we get

$$F(z) = -iP\sqrt{z^2 - c^2}\; G(z) \quad , \tag{31}$$

which is the solution corresponding to $F_\infty(z) = -iPz$. The crack tip stress intensity ration K_3/K_3^*, corresponding to (31), is

$$K_3/K_3^* = G(c) \quad . \tag{32}$$

Figure 3 shown the variation of crack tip stress intensity ratio as a function of a dimensionless

inclusion boundaries and dividing the result by its value at infinity.

In particular, taking $h(z) = -iP$ yields the solution corresponding to $F_\infty(z) = -iPz$. The crack tip stress intensity factor in this case is given by

$$K_3 = g(c)\,K_3^* \quad , \tag{35}$$

where K_3^* is given by (14) and $g(z)$ is $g(z)$ evaluated at the crack tip. If the crack tip touches the rigid inclusion, $g(c) = 0$ and, hence $K_3 = 0$. A similar result follows in the general case given by (34).

If $F_\infty(z) = \overline{F}_\infty(z)$; the crack does not influence the stress field.

The generalization of the present results to the case of inclusions on and off the x-axis is not obvious. It can be argued that the solution of this problem can be represented in the form (34) where

(i) $g(z)$ is the infinite product of all successive inversions of the crack singularity with respect to the inclusions with centers on the x-axis divided by its value at $z = \infty$; and

(ii) $h(z)$ is a function of type I that is imaginary on the boundaries of the inclusions with centers on the x-axis and such that $F(z)$ satisfies the boundary conditions on all other inclusions.

This is the expected representation of the solution for the case in which the crack influences the stress field. If the crack does not influence the stress field, the solution can be obtained rather easily by the Schwarz alternating method.

It should be noted that for an applied field of type I, the crack tip stress intensity factor will be given by

$$K_3 = i\sqrt{c}\,h(c)\,g(c) \quad , \tag{36}$$

where $h(c)$ and $g(c)$ are the functions described above evaluated at the crack tip. If the crack tip touches the rigid inclusion boundary, $g(c) = 0$ and K_3 will be zero.

6. CONCLUSION

It can be concluded from the results, presented above, that the crack tip stress intensity factor K_3 is identically zero for a mode three crack terminating radially at a rigid fiber independently of the fiber distribution. Since it is highly unlikely that a microcrack will always either miss the inclusions or run into a fiber in such a manner that the crack is normal to the interface, any interpretation of fracture phenomena in terms of the present results can be expected to be questionable. Analyses of non-radial cracks that will run into the interface at an oblique angle must be obtained before attempting to explain the fracture behavior of composites.

It is hoped that problems of non-radial cracks near inclusions can be solved in the near future. At that time a rational discussion of the micromechanics aspects of fracture of a single ply of a fiber reinforced composite will be possible.

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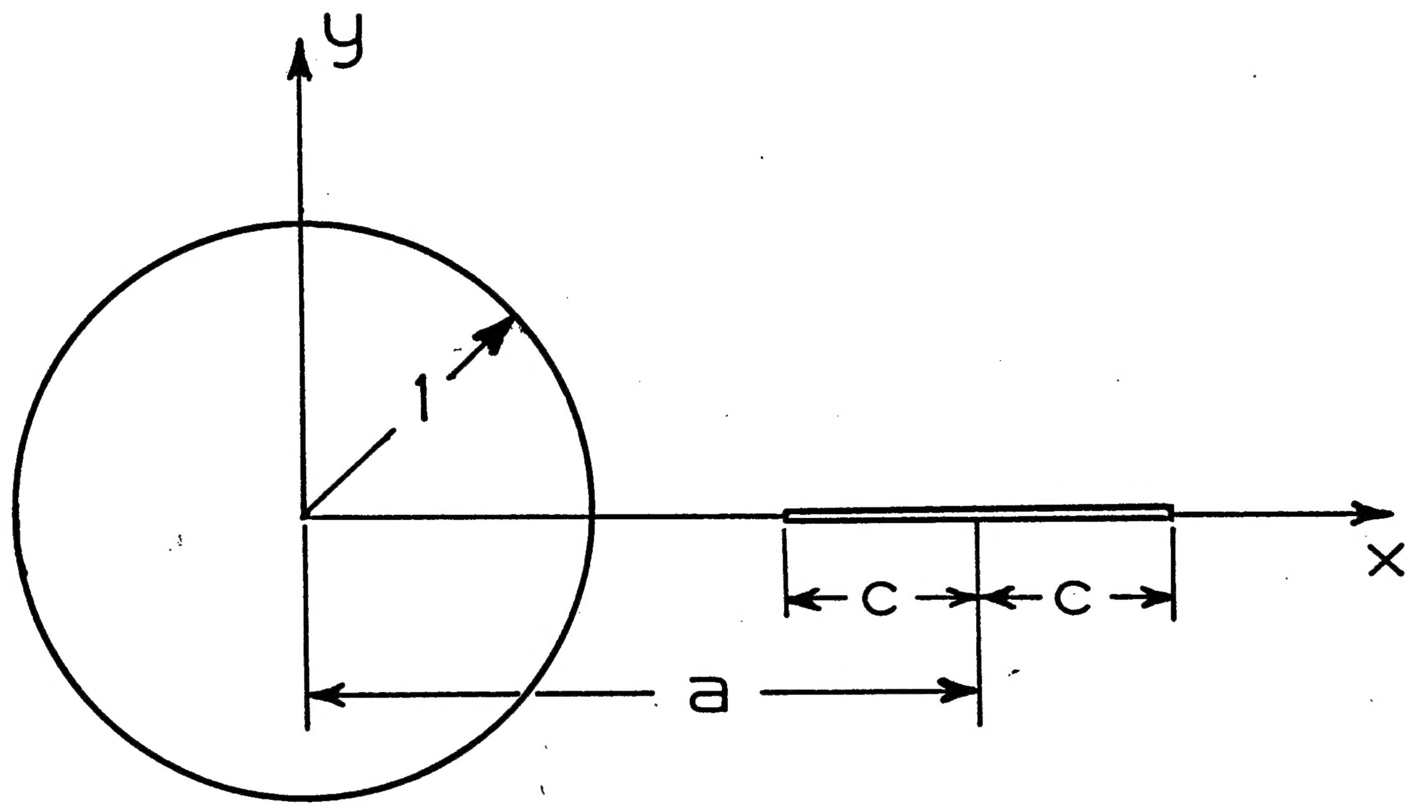
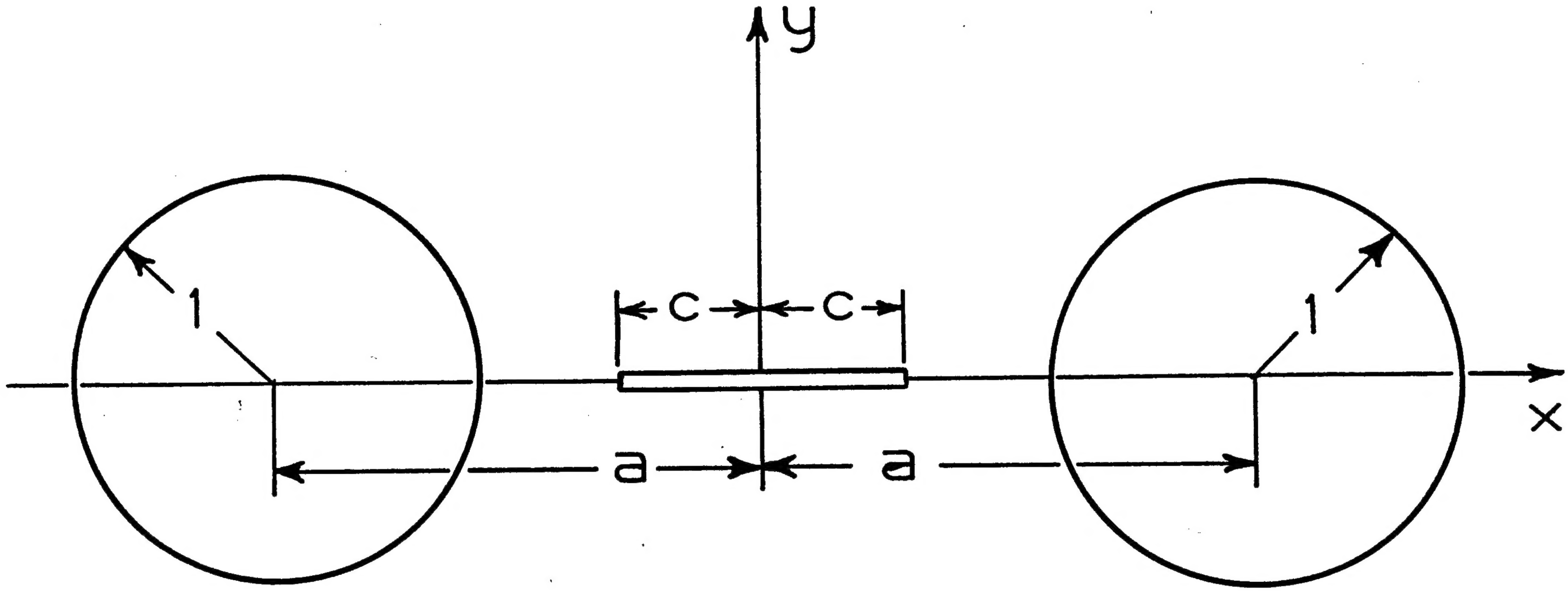
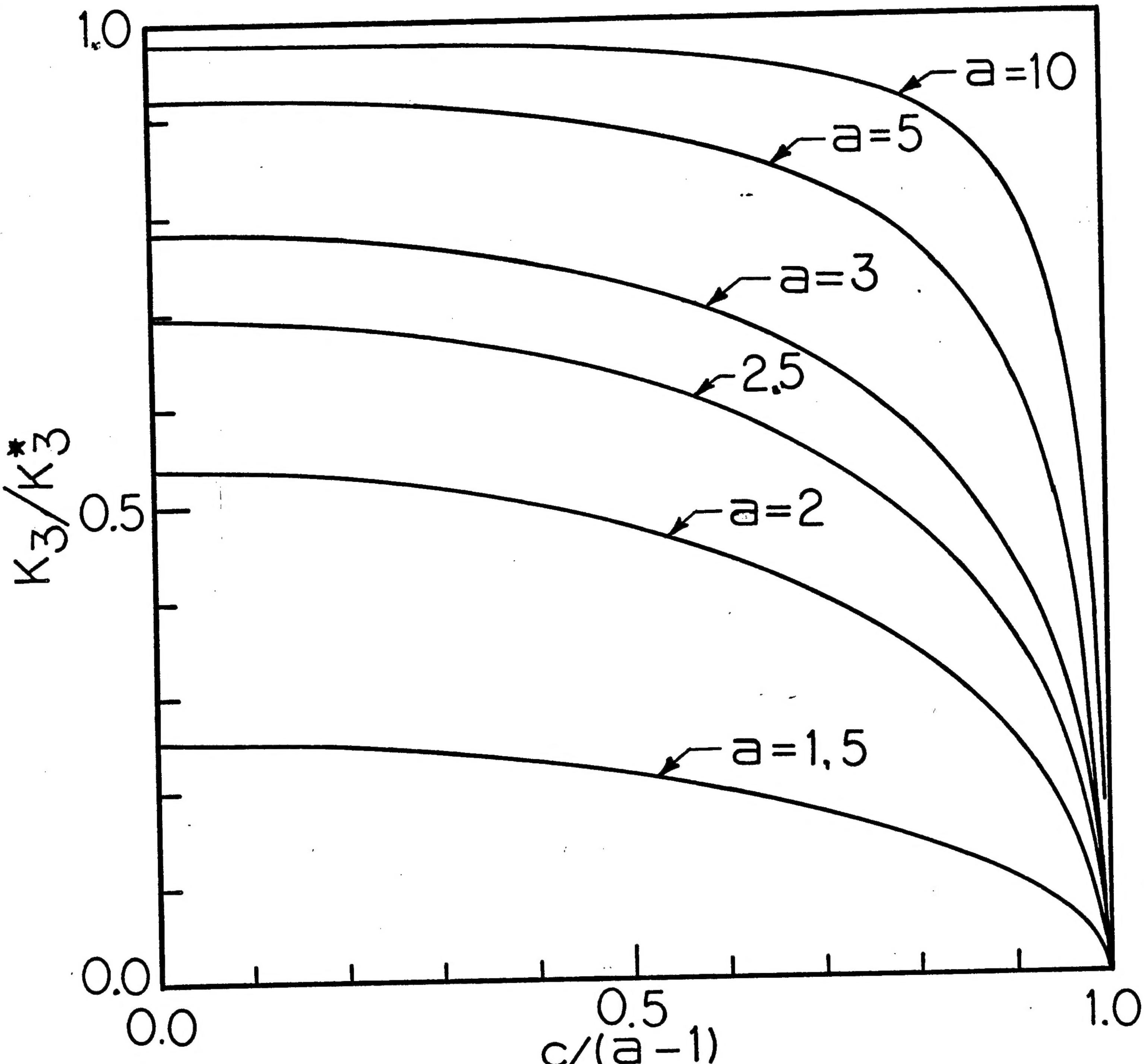
LIST OF FIGURES



Sendeckyj - Figure 1



Sendeckyj - Figure 2



Sendeckyj - Figure 3